Basic Properties of Subharmonic Injection Locking

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***Abstract*—Basic properties of subharmonic injection locking, as well as differences between subharmonic and fundamental injection locking, are studied with the describing function technique.**

The consequences of including a nonlinear capacitance as well as linear frequency-dependent parameters in the active element is investigated. The importance of the broad-band characteristics of the tuning circuit is studied. The detailed analysis is given for locking to the third subharmonic, but necessary expressions for generalizing the treatment to other than third subharmonic locking are provided. Finally, a note on stability of subharmonic locking is given.

I. Introduction

SUBHARMONIC phase locking of oscillators employing negative conductance elements such as avalanche and Gunn diodes has received much attention during the last few years [1]–[3]. The advantage of this technique over fundamental phase locking is obvious—varactor upmultiplication of the locking signal can be reduced or eliminated.

It has been shown possible to lock oscillators with an injected signal, the frequency of which is a submultiple of the free-running frequency. Earlier theoretical explanations of the behavior of subharmonically locked oscillators have mainly been based on an Adler type of treatment, using Van der Pol's equation as a starting point [4]. In a recent paper [5], subharmonic locking was treated in some detail, but that paper contains fundamental errors that led the authors to incorrect conclusions.

Subharmonic phase locking is, of course, related to harmonic phase locking. The problem of harmonically locked double-tuned oscillators has been studied by Cullen [6] and Markowski [7]. Some of our results in Sections V and VI therefore parallel some of those given in [6], [7].

The purpose of this paper is to study in detail a few cases of subharmonic locking. In this context, the basic properties of subharmonic phase locking as well as the basic differences between fundamental and subharmonic phase locking will be discussed.

II. Subharmonic Phase Locking: General Expressions

The oscillator is represented by the model shown in Fig. 1. The tuning circuit that is assumed to be nondissipative is represented by its voltage–current transmission matrix. With the notations of Fig. 1 we have

$$\begin{pmatrix} V_{1k} \\ I_{1k} \end{pmatrix} = \begin{pmatrix} T_{11}(k\omega) & T_{12}(k\omega) \\ T_{21}(k\omega) & T_{22}(k\omega) \end{pmatrix} \begin{pmatrix} V_{2k} \\ I_{2k} \end{pmatrix} \qquad (1)$$

where the index k denotes the kth frequency component (these components must not necessarily be harmonically related).

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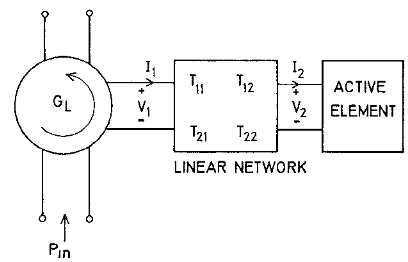

Fig. 1. Circulator-coupled negative nonlinear element.

G_L is the load of the oscillator and P_{in} is the injected power. The nonlinear active element is represented by its describing functions [8], denoted N_k, one for each frequency component. In this study the describing functions will have the dimension mho; they can be regarded as nonlinear admittances of the active element. It is easily seen [8] that the stationary state of the oscillator is determined by the following theoretically infinite set of equations:

$$N_1 \frac{G_L T_{12}(\omega)+T_{22}(\omega)}{G_L T_{11}(\omega)+T_{21}(\omega)} + 1 = \frac{I_e}{A_1} e^{j\theta} \frac{1}{G_L T_{11}(\omega)+T_{21}(\omega)} \qquad (2)$$

$$N_k \frac{G_L T_{12}(k\omega)+T_{22}(k\omega)}{G_L T_{11}(k\omega)+T_{21}(k\omega)} + 1 = 0, \qquad k \neq 1 \qquad (3)$$

where A_1 is $|V_{21}|$, I_e is a current source that is related to P_{in} [see (24)], and θ is the phase angle between that current source and V_{21}.

Equations (2) and (3) will be developed in detail in the following sections; they form the basis for the entire study.

III. Locking to the Third Subharmonic: The Single-Tuned Oscillator

In this section we study an oscillator, the tuning circuit of which is a simple parallel LC circuit. In this case,

$$G_L T_{12}(k\omega) + T_{22}(k\omega) = 1 \qquad (4)$$

$$G_L T_{11}(k\omega) + T_{21}(k\omega) = G_L + j(k\omega C - 1/k\omega L). \qquad (5)$$

The nonlinear active element is first assumed to obey the relation

$$i(t) = a_1 v(t) + a_3 v^3(t). \qquad (6)$$

The consequences of not including a second-order term, i.e., $a_2 v^2(t)$, are discussed below.

When locking to the third subharmonic we assume that only two frequency components: ω, which is the frequency of the injected signal, and 3ω, where $3\omega \approx \omega_0$, the free-running frequency, are present in $v(t)$. Considering the damping properties of normal tuning circuits, this is a realistic assumption.

We can thus write

$$v(t) = A_1 \sin(\omega t + \phi_1) + A_3 \sin 3\omega t \qquad (7)$$

and easily obtain the describing functions (see [9], [10])

$$N_1 = a_1 + \frac{3}{4} a_3(A_1^2 - A_1 A_3 e^{-j3\phi_1} + 2A_3^2) \qquad (8)$$

$$N_3 = a_1 + \frac{3}{4} a_3 \left(2A_1^2 - \frac{1}{3}\frac{A_1^3}{A_3} e^{j3\phi_1} + A_3^2\right). \qquad (9)$$

The qualitative differences between fundamental and subharmonic phase locking are easily illustrated by writing

$$(a_1 + \tfrac{3}{4}a_3 A^2) \frac{1}{G_L + j(3\omega C - 1/3\omega L)} + 1 = \frac{I_e}{A} e^{j\theta} \frac{1}{G_L + j(3\omega C - 1/3\omega L)} \qquad (10)$$

for fundamental locking, where power is injected at 3ω and no component is present at ω and

$$[a_1 + \tfrac{3}{4}a_3(A_1^2 - A_1 A_3 e^{-j3\phi_1} + 2A_3^2)] \frac{1}{G_L + j(\omega C - 1/\omega L)} + 1 = \frac{I_e}{A_1} e^{j\theta} \frac{1}{G_L + j(\omega C - 1/\omega L)} \qquad (11)$$

$$\left[a_1 + \frac{3}{4} a_3\left(2A_1^2 - \frac{1}{3}\frac{A_1^3}{A_3} e^{j3\phi_1} + A_3^2\right)\right] \cdot \frac{1}{G_L + j(3\omega C - 1/3\omega L)} + 1 = 0 \qquad (12)$$

for subharmonic locking.

The describing function $a_1 + (3/4)a_3A^2$ in (10) does not qualitatively change when power is injected at 3ω. The imaginary part in the left-hand side (LHS) of (10), introduced by the tuning circuit at frequencies $3\omega \neq \omega_0$, is balanced by the term in the right-hand side (RHS), representing the injected power. The describing function in (12), on the other hand, changes qualitatively as power is injected at ω, and thereby A_1 is introduced. Two terms containing A_1 are added; it is the complex term that balances the imaginary component from the tuning circuit. The balancing terms in (10) and (12) are thus qualitatively different, and this will establish a basic difference between fundamental and subharmonic locking.

We also see qualitatively that fundamental locking is more efficient for small injected powers than subharmonic locking—the "balancing" term in (10) contains an amplitude I_e related to the injected power, whereas the "balancing" term in (12) contains an amplitude A_1 related, although not directly, to the injected power, raised to the third order. Quantitative results will be given below.

There is another feature of subharmonic locking evident from (11) and (12): if I_e, and thereby A_1, is increased to enlarge the locking bandwidth, the amplitude A_3 at the desired output frequency 3ω will decrease. Quantitative results will be given below.

Having thus pointed out some basic properties of subharmonic locking we will give a quantitative analysis. In order to include the influence of capacitive nonlinearities we introduce a term $dv^3(t)/dt$; this is the least-order term that will influence locking to the third subharmonic:

$$i(t) = a_1 v(t) + a_3 v^3(t) + c_3 \frac{dv^3(t)}{dt}\cdot \qquad (13)$$

The a_3 in (8) and (9) are replaced by $a_3 + j\omega c_3$ and $a_3 + j3\omega c_3$, respectively. The same alterations are made in (11) and (12). The linear capacitance of the active element is included into the tuning circuit capacitance.

For a free-running oscillator, i.e., A_1 (and I_e) = 0, we have, with $3\omega = \omega_0$,

$$a_1 + \tfrac{3}{4}(a_3 + j\omega_0 c_3)A_{30}^2 + G_L + j\omega_0 C \cdot \left(1 - \frac{1}{\omega_0^2 LC}\right) = 0. \qquad (14)$$

If we introduce $g_0 = -a_1/G_L$, $g_2 = (3/4)a_3/G_L$, $b_2 = 3c_3/4G_L$, and $\omega_{00}^2 = 1/LC$ we find

$$A_{30}^2 = \frac{g_0 - 1}{g_2} \qquad (15)$$

and

$$\omega_0^2 = \frac{\omega_{00}^2}{1 + \frac{b_2 G_L}{C} A_{30}^2}\cdot \qquad (16)$$

If $b_2((g_0 - 1)G_L/g_2) \ll C$, the free-running frequency is close to ω_{00}.

If we remember that for normal phase locking of oscillators $|\Delta\omega| \ll \omega_0$, where $3\omega - \omega_0 = \Delta\omega$, (11) and (12) (with c_3 included) may be written as

$$g_2 A_{30}^2 = \frac{I_e}{A_1 G_L} \cos\theta$$

$$-Q_0' \frac{8}{3} = \frac{I_e}{A_1 G_L} \sin\theta \qquad (11b)$$

and, with $A_3^2 = A_{30}^2 + \Delta A_3^2$,

$$g_2 \Delta A_3^2 + 2g_2 A_1^2 + \frac{1}{3}\frac{A_1^3}{A_{30}} \cdot (-g_2 \cos 3\phi_1 + \omega_0 b_2 \sin 3\phi_1) = 0$$

$$2\frac{\Delta\omega}{\omega_0} Q_0 - \frac{1}{3}\frac{A_1^3}{A_{30}} (g_2 \sin 3\phi_1 + \omega_0 b_2 \cos 3\phi_1) + \omega_0 b_2(2A_1^2 + \Delta A_3^2) = 0 \qquad (12b)$$

where

$$Q_0 = \frac{\omega_0 C}{G_L}\left(1 + \frac{b_2 G_L}{C} A_{30}^2\right) \qquad (17)$$

$$Q_0' = \frac{\omega_0 C}{G_L}\left(1 + \frac{7}{8}\frac{b_2 G_L}{C} A_{30}^2\right) \qquad (18)$$

and Q_0 is the effective external Q value for the free-running oscillator. From (11b) and (15) we obtain

$$A_1 = \frac{I_e}{G_L Q_0' \frac{8}{3}} \frac{1}{\sqrt{1 + \left(\frac{3}{8Q_0'}(g_0 - 1)\right)^2}} \tag{19}$$

or, since $g_0 - 1 \approx 1$ (for an optimum loaded free-running oscillator $g_0 - 1 = 1$) and Q_0 is typically of an order 5–10,

$$A_1 \approx \frac{I_e}{G_L\left(Q_0' \frac{8}{3}\right)} \tag{19b}$$

which also means that the phase angle θ is close to $-90°$.

We are generally interested in the locking bandwidth, i.e., twice the maximum frequency deviation $(3\omega - \omega_0)$ obtainable. From (12b) we find that this requires $3\phi_1 = \pm 90°$ (which can be found by substituting ΔA_3^2 obtained from the first equation in (12b) into the second and calculating the maximum of $\Delta\omega$ with respect to $3\phi_1$, keeping A_1 constant). We then obtain at $\Delta\omega_{max}$,

$$\Delta A_3^2 = -2A_1^2 \mp \frac{1}{3} \frac{A_1^3}{A_{30}} \frac{\omega_0 b_2}{g_2} \approx -2A_1^2 \tag{20}$$

$$\frac{\Delta\omega_{max}}{\omega_0} = \pm \frac{1}{6Q_0} \frac{A_1^3}{A_{30}} \left(\frac{g_2^2 + (\omega_0 b_2)^2}{g_2}\right). \tag{21}$$

Equation (21) shows that the frequency deviation obtainable by subharmonic locking is symmetrical around ω_0 in contrast to the result in [5], where, however, the change in amplitude at 3ω was neglected. We see that by setting $\Delta A_3^2 = 0$ in (12b) we would also obtain an unsymmetrical frequency deviation. Another source of error in [5] is the deviation of [5, eq. (1)], which is valid only for the fundamental frequency and thus cannot be used when two frequencies exist simultaneously; thus, e.g., [5, eq. (6)] is incorrect.

Here we draw the conclusion that the nonlinear capacitance does not cause an unsymmetrical locking range (for small injected powers) as claimed in [5], but only changes the effective Q values of the circuit.

We note that (20) is approximately valid also for $|\Delta\omega| < |\Delta\omega_{max}|$ and combining (19b) and (20) we obtain an expression for the decrease in RF amplitude close to ω_0, when the injected power is increased:

$$\Delta A_3^2 \approx -2 \frac{I_e^2}{G_L^2\left(Q_0' \frac{8}{3}\right)^2}. \tag{22}$$

It was given above as a qualitative result that the amplitude A_3 will decrease if the injected current I_e is increased. The mechanism behind this can be briefly explained as follows: as power is injected at ω, and thereby the amplitude A_1 is introduced, this amplitude will give a positive contribution to the conductance of the active element at 3ω. This positive contribution will necessitate a decrease in the amplitude at 3ω in order that the total conductance of the active element at 3ω remains constant, and thereby an oscillation at 3ω can be maintained. This is true as long as A_1 is small so that the cubic term in A_1 can be neglected.

From (19b) and (21) we obtain

$$\left|\frac{\Delta\omega}{\omega_0}\right|_{max} = \frac{I_e^3(g_2^2 + (\omega_0 b_2)^2)}{6Q_0\left(Q_0' \frac{8}{3}\right)^3 G_L^3 g_2 A_{30}}. \tag{23}$$

For a circulator-coupled oscillator, the equivalent injected current I_e is related to the injected power P_{in} by

$$I_e = \sqrt{8P_{in}G_L} \tag{24}$$

and we get

$$\left|\frac{\Delta\omega}{\omega_0}\right|_{max} = \frac{9}{128Q_0(Q_0')^3}(g_0 - 1) \cdot \left(1 + \frac{(\omega_0 b_2)^2}{g_2^2}\right)\left(\frac{P_{in}}{P_0}\right)^{3/2}. \tag{25}$$

In [5] a similar expression was derived with another, incorrect, numerical coefficient.

For comparison, we give those expressions for fundamental locking that correspond to (22) and (25); they are, from [11], [12]

$$\Delta A^2 = \mp \frac{I_e}{G_L A_0} \frac{\omega_0 b_2}{g_2} \cdot \frac{1}{\sqrt{g_2^2 + \omega_0^2 b_2^2}} \tag{26}$$

$$\frac{\Delta\omega_{max}}{\omega_0} = \pm \frac{\sqrt{g_2^2 + (\omega_0 b_2)^2}}{Q_0 g_2}\left(\frac{P_{in}}{P_0}\right)^{1/2}. \tag{27}$$

As seen from (26) there is no change in RF voltage if the nonlinear capacitance is absent, i.e., $b_2 = 0$ [(26) and (27) are, of course, valid only for small frequency deviations].

In those cases where the subharmonic power is not injected through a circulator, but, e.g., like in [13] through the bias circuit, an appropriate relation between I_e and the injected power P_{in} should replace (24).

Finally, it should be noticed that the unsymmetrical locking properties seen experimentally [5] cannot be explained by assuming a more general tuning network; they can, however, be accounted for by including a second-order term $a_2v^2(t)$ in the nonlinear active element and including mixed frequencies at 2ω and 4ω; in fact, a nonlinear capacitance is not necessary for obtaining unsymmetrical locking properties. Another possible explanation is the use of a nonideal dc source (i.e., a dc-bias circuit having $R \neq 0$). These problems will be dealt with in [14].

IV. Locking to the Third Subharmonic: Broad-Band Properties of the Active Element

We have so far employed a rather special nonlinear device. Generally,

$$i(t) = f\left(v, \frac{dv}{dt}, \frac{d^2v}{dt^2}, \cdots, \frac{d^nv}{dt^n}\right)$$

which gives rise to frequency dependent conductances and capacitances (in case the model should include nonlinear inductances, integrals of v should be introduced into the argument of f). After a series expansion of $i(t)$, the describing functions can be calculated, and thus this general case may also be dealt with. The frequency dependence of the device characteristic is important for the subharmonic locking properties, and to demonstrate this we will treat here a very simple case, namely a frequency-dependent linear part of the conductance and capacitance of the active element, which is given by the first-order terms in even and odd derivatives, respectively. Since we treat a linear term no mixing between frequencies will occur, which means that we can introduce different linear conductances and capacitances on ω and 3ω.

At 3ω we assume a_1 and C as before and at ω:

$$a_1(\omega) = \alpha G_L \quad \text{and} \quad C(\omega) = \gamma C \tag{28}$$

where α and γ describe the frequency dependence (the capacitance of both the active element itself and of the tuning circuit are incorporated into C).

If we insert this into the describing functions for 3ω and ω, assuming $b_2=0$ (i.e., neglecting the nonlinear capacitance), we find that the only change is in A_1, which now is given by

	$3\frac{1}{3}$ GHz	10 GHz
conductance mmho	≈ -0.1	≈ -0.9
susceptance mmho	≈ -25	≈ 2.5

If we want $g_0=2$, we must choose $G_L=0.45\cdot 10^{-3}$ and we then obtain $Q_0=5.6$. From the above table we calculate $\alpha\approx -0.2$ and $\gamma\approx -30$.

If we insert these values into (30) we see that the locking bandwidth is about 2 percent of that obtained for a circuit where the small signal parameters are considered to be constant.

This figure could of course be increased by using a tuning circuit with a capacitance that dominated over the device capacitance, but this would be of no value since the Q_0 value of the oscillator would also increase, and therefore the locking bandwidth would decrease.

The frequency dependence of the device parameters is of

$$A_1 = \frac{I_e}{G_L \frac{Q_0|9-\gamma|}{3}\sqrt{1+\frac{1}{Q_0^2}\left(\frac{3}{9-\gamma}\right)^2(2g_0+\alpha-1)^2}} \tag{29}$$

and hence the locking bandwidth will be

$$K = \frac{8^3\left(\sqrt{1+\left(\frac{3}{8Q_0}\right)^2(g_0-1)^2}\right)^3}{|9-\gamma|^3\left(\sqrt{1+\left(\frac{3}{(9-\gamma)Q_0}\right)^2(2g_0+\alpha-1)^2}\right)^3} \tag{30}$$

of that obtained for a frequency-independent conductance and capacitance. Fig. 2 shows the change in normalized locking bandwidth as a function of α for two different Q_0 values with $\gamma=1$ and $g_0=2$. Fig. 3 shows the normalized locking bandwidth versus γ with α as a parameter ($Q_0=10$ and $g_0=2$). Figs. 2 and 3 show that the frequency dependence of the admittance of the active element can change the locking bandwidth considerably. In order to calculate the largest possible consequences that this frequency dependence can have, we assume for the remainder of this section that the tuning circuit is purely inductive. As a practical example we take an IMPATT diode that has a small signal admittance given in the Read approximation as [15]

$$Y_{ss} = \frac{\omega C_d \frac{1-\cos\theta_d}{\theta_d}\cdot\frac{1}{1-\omega^2/\omega_a^2} + j\omega C_d\left(1-\frac{1}{1-\omega^2/\omega_a^2}\frac{\sin\theta_d}{\theta_d}\right)}{1+\frac{2}{\theta_d^2}\cdot\frac{1}{1-\omega^2/\omega_a^2}\left(\frac{1}{1-\omega^2/\omega_a^2}-\frac{\cos\theta_d}{1-\omega^2/\omega_a^2}-\theta_d\sin\theta_d\right)} \tag{31}$$

where ω_a is the avalanche frequency, θ_d is the drift angle, and C_d is the geometrical capacitance of the drift region. Assuming $\theta_d(3\omega)=\pi$, $3\omega=1.2\,\omega_a$, and $g_0=2$, we find $\alpha=7.5$, $\gamma=0.44$, and a locking bandwidth 67 percent of that obtained using a frequency-independent device admittance.

As another even more striking example, we can extrapolate slightly the measurements of the small signal parameters for a p-i-n diode, made by Josenhans and Misawa [16][1] to yield the following:

[1] The extrapolations were made from curves published in *Microwave Semiconductor Devices and Their Circuit Applications*, H. A. Watson, Ed. New York: McGraw-Hill, 1968.

little importance for the fundamental locking properties, and this is a basic difference between fundamental and subharmonic phase locking. It should also be noted that the characteristics of the circulator are likely not to be the same at ω as at 3ω. These differences could be treated in a way analogous to the above; as a matter of fact, the frequency changes in both the circulator and the active element can be incorporated into the α and γ parameters introduced in (28).

V. Locking to the Third Subharmonic: The Double-Tuned Oscillator

The single-tuned oscillator treated in Sections III and IV is of course the simplest possible oscillator. We will now investigate the advantage in locking properties that can be obtained by shaping the linear tuning circuit. We will employ a double-tuned circuit, resonant close to ω as well as close to 3ω (see Fig. 4). For the purpose of this section it is advantageous to use the simplest possible nonlinear active element, as defined in (6).

For small deviations from the free-running frequency we have

$$G_L T_{12}(\omega) + T_{22}(\omega) = G_L T_{12}(3\omega) + T_{22}(3\omega) = 1 \tag{32a}$$

$$G_L T_{11}(\omega) + T_{21}(\omega) \approx G_L + j(\omega C_1 - 1/\omega L_1) \tag{32b}$$

$$G_L T_{11}(3\omega) + T_{21}(3\omega) \approx G_L + j(3\omega C_3 - 1/3\omega L_3) \tag{32c}$$

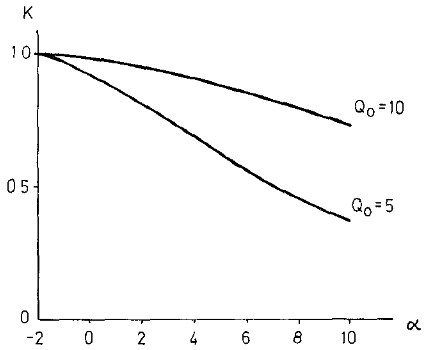

Fig. 2. Normalized locking bandwidth K versus α; α is defined by (28) and K by (30). α describes the frequency dependence of the linear part of the device conductance. A frequency-independent linear capacitance ($\gamma=1$) and $g_0=2$ were assumed.

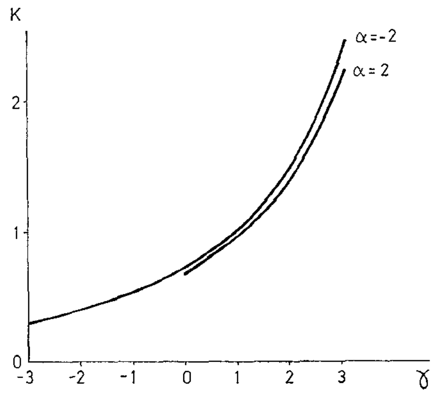

Fig. 3. Normalized locking bandwidth K versus γ for two α values: $Q_0=10$ and $g_0=2$.

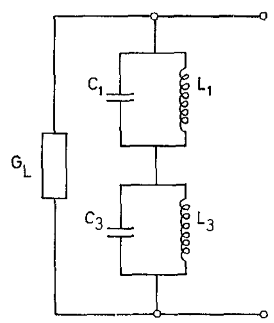

Fig. 4. Double-tuned circuit $\omega_{10}^2=1/L_1C_1$; $3^2\omega_{10}^2=1/L_3C_3$.

where

$$\omega^2 \approx \omega_{10}^2 = \frac{1}{L_1C_1} \quad \text{and} \quad 9\omega^2 \approx \omega_{30}^2 = \frac{1}{L_3C_3}.$$

Furthermore, we have to observe that in order to avoid hopping effects or multiple-frequency oscillations, the circuit should be passive at the frequency ω. Since a practical device often has a frequency-dependent negative conductance (compare an IMPATT diode) this is a fair assumption. Note that for a single-tuned oscillator we did not have to make this assumption since the linear network anyhow strongly damped any oscillations on the frequency ω. For simplicity we assume

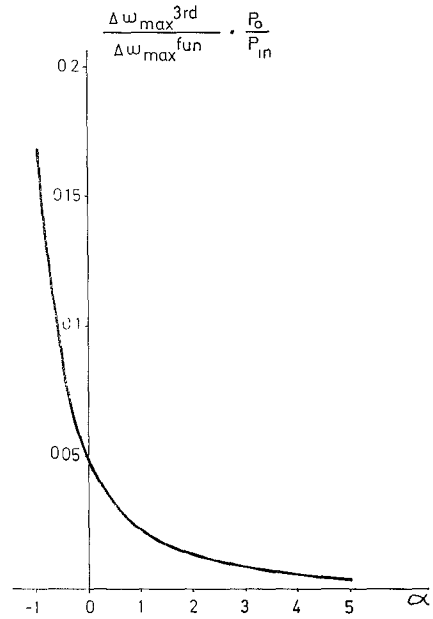

Fig. 5. Subharmonic locking bandwidth (third subharmonic) for a double-tuned circuit divided by fundamental locking bandwidth versus α assuming the same amount of injected power in the two cases. The curve shown is normalized by (P_{in}/P_0). Furthermore, $g_0=2$ was assumed.

that the conductance of the active element on frequency ω is αG_L $(\alpha > -1)$.

If we assume $A_1 \ll A_{30}$ we find in the same way as before that

$$A_1 = \frac{I_e}{G_L(2g_0 - 1 + \alpha)} \tag{33}$$

and that $\cos\theta \approx 1$, i.e., $\theta = 0°$ in this case (compare $\sin\theta \approx -1$ for the single-tuned circuit). A_1 given by (33) may be inserted directly into (21) setting $b_2=0$. A more accurate expression than (19) for A_1 in a single-tuned oscillator, with a_1 replaced by αG_L, is the following:

$$A_1 = \frac{I_e}{G_LQ_0 \frac{8}{3}\left[1 + \left(\frac{3}{Q_0 8}\right)^2 (2g_0 + \alpha - 1)^2\right]^{1/2}} \tag{19c}$$

where we have chosen $b_2=0$.

We find from (21) by comparing (33) with (19c) that

$$\left|\frac{\Delta\omega}{\omega_0}\right|_{\max}^{\text{double tuned}} = \left[\left(\frac{Q_0 \frac{8}{3}}{2g_0 + \alpha - 1}\right)^2 + 1\right]^{3/2} \cdot \left|\frac{\Delta\omega}{\omega_0}\right|_{\max}^{\text{single tuned}}. \tag{34}$$

Equation (34) shows the gain in locking bandwidth obtained with a double-tuned circuit. It is seen that the larger the passive conductance at the subharmonic frequency (i.e., the larger α), the smaller the gain.

It is obvious that a double-tuned circuit increases the locking bandwidth appreciably at subharmonic locking for a given injected power P_{in}.

It is also interesting to compare locking to the third subharmonic through a double-tuned circuit with fundamental locking, assuming the same amount of injected power in the two cases. From (27) and (33), (21) and (15), respectively, we obtain, with $b_2=0$,

$$\frac{\Delta\omega_{max}{}^{3rd\ sub}}{\Delta\omega_{max}{}^{fun}} = \frac{4(g_0-1)}{3(2g_0+\alpha-1)^3}\left(\frac{P_{in}{}^{3rd}}{P_0}\right)^{3/2} \cdot \left(\frac{P_0}{P_{in}{}^{fun}}\right)^{1/2} \qquad (35)$$

where

$$(P_{in}{}^{3rd}/P_0)^{3/2}(P_0/P_{in}{}^{fun})^{1/2} = \frac{P_{in}}{P_0}$$

if the same amount of injected power is assumed in the two cases. Equation (35) is illustrated in Fig. 5.

VI. Locking to the *n*th Subharmonic

The basic properties of subharmonic locking which were discussed in the preceding sections for the case $n=3$ also hold for $n>3$; they are only more pronounced.

Assuming an active element obeying the relation

$$i(t) = \sum_{k=1}^{n} a_k v^k(t) \qquad (36)$$

and assuming

$$v(t) = A_1 \sin(\omega t + \phi_1) + A_n \sin n\omega t \qquad (37)$$

we employ the binomial theorem and the formulas for expansion of $\sin^n x$ and $\cos^n x$ [17] to find the describing functions

$$N_1 = \sum_{k=1}^{n} a_k \sum_{\substack{p=0 \\ p\ \text{even}}}^{k-1} \frac{k!\left(\frac{1}{2}\right)^{k-1} A_1{}^{k-p-1} A_n{}^p}{\left(\frac{k-p+1}{2}\right)!\left(\frac{k-p-1}{2}\right)!\left(\frac{p}{2}\right)!\left(\frac{p}{2}\right)!} + a_n \frac{n(-1)^{(n-1)/2}}{2^{n-1}} A_1{}^{n-2} A_n e^{-jn\phi_1} \qquad (38)$$

$$N_n = \sum_{k=1}^{n} a_k \sum_{\substack{p=1 \\ p\ \text{odd}}}^{k} \frac{k!\left(\frac{1}{2}\right)^{k-1} A_1{}^{k-p} A_n{}^{p-1}}{\left(\frac{k-p}{2}\right)!\left(\frac{k-p}{2}\right)!\left(\frac{p+1}{2}\right)!\left(\frac{p+1}{2}\right)!} + a_n \frac{(-1)^{(n-1)/2}}{2^{n-1}} \frac{A_1{}^n}{A_n} e^{jn\phi_1}. \qquad (39)$$

For small injected power, i.e., A_1 small, we need to retain terms in N_n containing A_1 up to the second order only, plus the last term, since this is the only complex term, the balancing term when $n\omega \neq \omega_0$.

It is to be noted that if $a_n \approx 0$, no locking takes place unless frequency components in $v(t)$ other than ω and $n\omega$ are included in the analysis, introducing other complex terms in N_n. This locking mechanism is, however, extremely inefficient unless multiple-tuned circuits (e.g., resonant for ω, 3ω, and 9ω, if $n=9$) are used, and this possibility may therefore be neglected.

Expressions for maximum locking bandwidth, etc., can be obtained in much the same way as for the case $n=3$.

VII. Stability Considerations

Stability without restrictions regarding frequency deviation and injected power can be investigated in a way similar to that discussed for fundamental locking in [8], but with much greater numerical difficulties.

Here we will discuss stability for a case with small frequency deviation and small injected power; the oscillator is locked with the nth subharmonic. We first assume that a stationary solution is given by

$$N_n + G_L + j2\frac{\Delta\omega}{\omega_0}\frac{G_L}{Q_0} = 0. \qquad (40)$$

We introduce the notation ϕ_0 for $n\phi_1$ for this stationary solution. We then perturb the solution slightly so that at $t=0$ we have $\phi=\phi_0+\Delta\phi$. If we introduce this phase into (40) we will get an equation, the real part of which is almost identically the same as when $\phi=\phi_0$, but with an imaginary part that is substantially changed. We easily find the difference in the imaginary part, to the first approximation, introduced by $\Delta\phi$ to be given by

$$\frac{\partial\Delta\phi}{\partial t} = \frac{-a_n}{2^{n-1}Q_0G_L}\frac{A_1{}^n}{A_n}\omega_0(-1)^{(n-1)/2}\cos\phi_0\Delta\phi. \qquad (41)$$

If the original solution of (40) is to be stable, then evidently the perturbation is to decrease with time, and thus $\partial\Delta\phi/\partial t$ has to have the opposite sign of $\Delta\phi$. This leads to the condition

$$(-1)^{(n-1)/2}\cos\phi_0 > 0 \qquad (42)$$

for a stable solution, i.e.,

$$\frac{\pi}{2} < \phi_0 < \frac{3\pi}{2} \qquad (43)$$

for $n=3$, and

$$\frac{-\pi}{2} < \phi_0 < \frac{\pi}{2} \qquad (44)$$

for $n=5$.

VIII. Conclusions

Subharmonic injection locking is attractive since upmultiplication of the locking signal can be reduced in order or even avoided. There are, however, (as shown in this paper) basic differences between fundamental and subharmonic injection locking that make the latter less efficient. This makes subharmonic locking feasible for applications where bandwidth is of little importance. The possibility of shifting the free-running frequency (through varactor tuning) in approximate synchronization with the change in frequency of the injected signal has not been investigated for subharmonic injection locking in this paper; with such a method, injection locking would provide only a part of the locking mechanism and locking bandwidth could be increased. The theory presented in this paper would be valid for that part of the locking that can be attributed to injection locking.

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Microstrip Dispersion Model

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Abstract—**The assumption that the quasi-TEM mode on microstrip is primarily a single longitudinal-section electric (LSE) mode leads to a transmission line model whose dispersion behavior can be analyzed and related to that of microstrip. Appropriate approximations yield simple, closed-form expressions that allow slide-rule prediction of microstrip dispersion.**

Nomenclature

a, a', b, b', s, w	Mechanical dimensions of conventional microstrip and the LSE mode model (Fig. 2).
c	Speed of light in free space = 11.8 in/ns.
C'	Capacitance per unit length of microstrip line at zero frequency.
D	Width of the zero-frequency parallel-plate microstrip equivalent structure.
f	Frequency.
f_i	Frequency of inflection of the dispersion curve.
f_p	Parameter of the dispersion function.
G	Empirical parameter used to simplify the microstrip dispersion function.
k_o	Free-space wavenumber.
L'	Inductance per unit length of microstrip line at zero frequency.
Z_f	Microstrip characteristic impedance at frequency f.
Z_0	Microstrip characteristic impedance at zero frequency.
γ	Propagation constant along the microstrip line.
γ_a	Transverse propagation constant in the air-filled part of the microstrip model.
γ_s	Transverse propagation constant in the dielectric-filled part of the microstrip model.
ϵ_e	Microstrip effective dielectric constant (a function of frequency).
ϵ_{ei}	Microstrip effective dielectric constant at the inflection point.
ϵ_{e0}	Microstrip effective dielectric constant at zero frequency.
ϵ_o	Permittivity of free space = 8.85×10^{-12} F/m.
ϵ_s	Substrate relative dielectric constant.
η_o	Impedance of free space = 376.7 Ω.
μ_o	Permeability of free space = 31.92 nH/in, or $4\pi\times10^{-7}$ H/m.
ω	Radian frequency.

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